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Algorithm for Missile Detection from Radar Data

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I. Introduction

THE present paper is devoted to the problem of identification of the characteristics of a missile and of its trajectory on the basis of few radar measurements that are affected by rather large errors.

If radar measurements are accurate, it is proved that four observations are enough to give a precise estimate of the unknown parameters of the boost phase. For radar measurements corrupted by noise, the boost parameter estimates provided by the method proposed are used as a first guess for the solution of a maximum likelihood estimator (MLE), and the combined use of the parameters estimate and MLE allows the detection of missile characteristics and trajectory, even with rather large error measurements.

For some applications, it is important to have on-time tracking of the observed object. Then an extended Kalman filter (EKF) is applied to the present problem, and the performances of the method are found:

1) The EKF seems unable to identify all the missile parameters $\boldsymbol{p}_1$ and the state vector $\boldsymbol{X}_1$ on the basis of few observations. The convergence of the algorithm is possible if not-large variations of the nominal missile parameters are considered, as it is also shown in [1].

2) The EKF is successful in the determination of the state vector $\boldsymbol{X}_1$ that determines the trajectory in the ballistic phase.

A combined use of the two methods (parameter estimate/MLE in the boost phase and EKF in the subsequent phases) seems to be effective. Of course, it is important to know the switching time from one method to the other, and, in fact, the algorithm proposed here is able to detect the missile burnout time.

II. Missile Detection from Deterministic Data

The fundamental parameters of (each stage of) a missile are 1) initial thrust-to-weight ratio $n_0 = T/gm_0$ (dimensionless), 2) reduced ballistic coefficient $\beta_0 = \frac{1}{2}S/m_0$ (m^2/kg), 3) specific impulse I_{sp} (s), and 4) structural-over-total-mass ratio $u = m_s/m_0$ (dimensionless). Other missile parameters can be derived from the fundamental parameters: 1) relative mass rate $q_0 = \dot{m}/m_0 = n_0/I_{sp}$ (dimensionless), 2) burn time $t_b = (1-u)/q_0$ (s), 3) thrust-over-weight ratio $n(t) = n_0/(1-q_0 t)$ (dimensionless), and 4) ballistic coefficient $B = C_D[\beta_0/(1-q_0 t)]$ (m^2/kg). The missile equations of motion in an inertial reference frame system are

$$\begin{cases} \dot{X} = V_X \\ \dot{Y} = V_Y \\ \dot{Z} = V_Z \\ \dot{V}_X = -\frac{\mu}{r^3}X - \rho V_R \frac{\beta_0}{1-qt} V_{R_X} + g\frac{n_0}{1-qt} l_X \\ \dot{V}_Y = -\frac{\mu}{r^3}Y - \rho V_R \frac{\beta_0}{1-qt} V_{R_Y} + g\frac{n_0}{1-qt} l_Y \\ \dot{V}_Z = -\frac{\mu}{r^3}Z - \rho V_R \frac{\beta_0}{1-qt} V_{R_Z} + g\frac{n_0}{1-qt} l_Z \end{cases} \tag{1}$$

where $\boldsymbol{V} = (V_X, V_Y, V_Z)$ is the absolute velocity, $\boldsymbol{V}_R = (V_{RX}, V_{RY}, V_{RZ})$ is the relative-to-wind velocity $[\boldsymbol{V}_R = \boldsymbol{V} - \boldsymbol{V}_w$ (the wind velocity is assumed to be $\boldsymbol{V}_w = \boldsymbol{\omega}_E \times \boldsymbol{r})]$, $\rho = \rho_0 e^{-kh}$ is the exponential model of air density, and $g = 9.81$ m/s^2.

The thrust direction $\hat{T} = (l_X, l_Y, l_Z)$ depends on the different arches of the missile trajectory; during boost phase, a gravity-turn trajectory (zero incidence angle) is generally followed, hence,

$$\hat{T} = \hat{V} = \left(\frac{V_X}{V}, \frac{V_Y}{V}, \frac{V_Z}{V}\right)$$

It is supposed that at times $\tau_0 = 0, \tau_1, \ldots, \tau_{N-1}$, measures of the missile range and azimuth and elevation angles (ρ_k, Az_k, El_k), $k = 0, \ldots, N-1$ are taken by a radar station; in this section, the measures are not affected by errors and they detect the missile in gravity-turn trajectory. Then the observed trajectory is basically planar; the Earth center and the points P_1 and P_2, corresponding to the first two observed positions, determine the plane of the trajectory. This plane is the x–y plane where the $\hat{x}$ axis is on the radial direction $\boldsymbol{r}_1$ of the first observed point P_1. The radar data are transformed in x, y, and z coordinates; if the missile trajectory is exactly planar, the y coordinate will be equal to zero for all the observed positions and average velocities and flight path angle can be defined as

$$\boldsymbol{V}_k = \frac{\boldsymbol{r}(\tau_{i+1}) - \boldsymbol{r}(\tau_i)}{\tau_{i+1} - \tau_i}, \qquad \gamma_k = \operatorname{atan}\left(\frac{z(\tau_{i+1}) - z(\tau_i)}{x(\tau_{i+1}) - x(\tau_i)}\right) \tag{2}$$

These are $N-1$ trajectory data evaluated at time $\theta_k = \tau_k + (\Delta\tau/2)$ where $k = 0, N-2$.

The velocity and flight path angle data [Eq. (2)] allow an estimate of the thrust-to-weight ratio using the analytic solution of the gravity-turn trajectory proposed in [2]. Namely, if (V_{k-1}, ξ_{k-1}) and (V_k, ξ_k) are the velocity and the $\xi = \tan[(\pi/4) - (\gamma/2)]$ variable at time θ_{k-1} and θ_k, the thrust-over-weight ratio n_k (regarded as constant) satisfies the equation:

$$f(n_k) = V_k\left(1 + \xi_{k-1}^2\right)\xi_{k-1}^{n_k-1} - V_{k-1}\left(1 + \xi_k^2\right)\xi_k^{n_k-1} = 0 \tag{3}$$

The solution $\bar{n}_k$ of Eq. (3) is the average thrust-over-weight ratio value in the interval $[\theta_{k-1}, \theta_k]$, and so it can be assumed to be the value of n at τ_k.

Note that N observations generate $N-2$ estimated values $\bar{n}_k$. At least two increasing values $\bar{n}_k$ (hence, four observations with $\bar{n}_2 > \bar{n}_1$) are needed to approximate the variation with time of the thrust-over-weight ratio value:

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$$\bar{n}(\tau) = \frac{\bar{n}_1}{1 - \bar{q}_1(\tau - \tau_1)}, \qquad \tau \geq \tau_1 \tag{4}$$

where

$$\bar{q}_1 = \frac{[1 - (\bar{n}_1/\bar{n}_2)]}{\Delta\tau}$$

If more than two values of $\bar{n}_k$ are increasing, a best-fitting procedure is applied to the parameters $(\bar{n}_1, \bar{q}_1)$ of Eq. (4). Generally, two different situations may occur:

1) The values $\bar{n}_k$ obtained by the observation data are all increasing
2) There is a value $\bar{n}_j$ such that $\bar{n}_k < \bar{n}_j$ for $k = j + 1, \ldots, N - 2$

In case 1, all the observations are taken in the boost phase, and so it is difficult to foresee the burnout time; however, one can observe that Eq. (4) has an asymptote at time $\tau^* = (1/\bar{q}_1) + \tau_1$.

This is the time needed to burn all the mass of the missile as if it were composed of propellant only. In fact, there is a structural mass m_s, so that $\bar{u}_1 = m_s/m(\tau_1) \neq 0$ and

$$\tau_b = (1 - \bar{u}_1)/\bar{q}_1 = (1 - \bar{u}_1)(\tau^* - \tau_1)$$

In the preceding formula, the parameter $\bar{u}_1$ is unknown; because the value u at launch can be taken approximately equal to 0.2, a reasonable range of values of $\bar{u}_1$ is 0.3–0.8, and so a first estimate of τ_b is

$$0.2(\tau^* - \tau_1) \leq \tau_b \leq 0.7(\tau^* - \tau_1)$$

Such an estimate can be improved if the observations catch the ballistic phase of the missile, as in case 2. In such a case, one immediately has the estimate

$$\tau_j \leq \tau_b \leq \tau_{j+1}$$

A better estimate can be achieved integrating the planar equations of motion with the conditions at θ_0 obtained in Eq. (2), until time τ_{j+1} with burnout time $\tau = \tau_j + \mathrm{d}\tau$, where $\mathrm{d}\tau$ is variable in $(0, \Delta\tau)$. The result of the integration determines a pair (V_f, z_f), which, together with the pair (V_j, z_j), will define the value $\bar{n}_f$. There is a value $\mathrm{d}\tau^*$ such that $\bar{n}_f$ best approximates the "observed" value $\bar{n}_{j+1}$, and so the estimated burnout time is $\tau_b = \tau_j + \mathrm{d}\tau^*$.

To estimate the ballistic factor, note that in the formula

$$B = C_D \frac{S}{2m} = C_D \beta = C_D \frac{\beta_1}{1 - \bar{q}_1(\tau - \tau_1)}, \qquad \tau \geq \tau_1$$

C_D has known variation with the Mach number for a typical missile configuration; then it is enough to compute the reduced ballistic coefficient β_1 at observation time τ_1.

This is obtained by evaluation of the drag loss

$$V(\theta_2) - V_2 = \Delta V_a = \int_{\theta_{j1}}^{\theta_{j2}} \rho V^2 B \, \mathrm{d}\tau \tag{5}$$

where the left-hand side is the difference between the numerically integrated value of velocity and the observed value V_2, and in the right-hand side a linear dependence of the velocity V and altitude h in the time interval $[\theta_{j1}, \theta_{j2}]$ is assumed:

$$V(\tau) = V_{j1} + \frac{V_{j2} - V_{j1}}{\theta_{j2} - \theta_{j1}}(\tau - \theta_{j1})$$
$$h(\tau) = h_{j1} + \frac{h_{j2} - h_{j1}}{\theta_{j2} - \theta_{j1}}(\tau - \theta_{j1}) \tag{6}$$

Then Eq. (5) implies

$$\beta_1 = \frac{\Delta V_a}{\int_{\theta_{j1}}^{\theta_{j2}} C_D \rho_1 e^{-k[h(\tau) - h_1]} V(\tau)^2 / [1 - \bar{q}_1(\tau - \tau_1)] \, \mathrm{d}\tau} \tag{7}$$

where ρ_1 is the air density at h_{j1}, and $V(\tau)$ and $h(\tau)$ are defined in Eq. (6).

Table 1 Observation times, position coordinates, and estimated and real thrust-to-weight ratio

Time from launch and from first observation	x, km	y, km	z, km	$\bar{n}_k$	$n(t)$
$t_0 = 79$ s, $\tau_0 = 0$	6428.5	0	0	$\backslash \equiv$	5.23
$t_1 = 89$ s, $\tau_1 = 10$	6466.6	$7e - 13$	11.6	6.5	6.33
$t_2 = 99$ s, $\tau_2 = 20$	6494.0	$2e - 12$	26.0	8.3	8.03
$t_3 = 109$ s, $\tau_3 = 30$	6524.7	$3e - 12$	45.9	10.3	10.96
$t_4 = 119$ s, $\tau_4 = 40$	6549.8	$4e - 12$	68.9	1.5	0
$t_5 = 129$ s, $\tau_5 = 50$	6602.7	$6e - 12$	88.2	−0.02	0
$t_6 = 139$ s, $\tau_6 = 60$	6628.2	$7e - 12$	130.2	−0.03	0
$t_7 = 149$ s, $\tau_7 = 70$	6653.1	$8e - 12$	151.2	−0.04	0
$t_8 = 159$ s, $\tau_8 = 80$	6678.1	$9e - 12$	172.1	−0.04	0
$t_9 = 169$ s, $\tau_9 = 90$	6492.8	$1e - 11$	193.7	$\equiv$	0

As an example, a missile is launched from latitude $L_0 = 55°$ and sidereal longitude $\lambda_0 = -20°$ with parameters at launch time of

$$n_0 = 2.2, \qquad I_{sp} = 300 \text{ s}, \qquad u_0 = 0.17, \qquad \beta_0 = 4.15e - 05$$
$$q_0 = 7.33e - 03, \qquad t_b = 113 \text{ s}$$

A station sited at latitude $L_S = 54°$ and sidereal longitude at launch $\lambda_S = -15°$ is provided with a radar able to observe objects in the range of 5–60_deg of elevation angle. The radar in standby mode rotates with a velocity of one revolution in 10 s and observes the missile approaching after $t_0 = 79$ s from launch. Ten observations of the missile are taken. The missile parameters at time of first observation are

$$n_1 = 5.23, \qquad I_{sp} = 300 \text{ s}, \qquad u_1 = 0.40$$
$$\beta_1 = 9.86e - 05, \qquad q_1 = 1.74e - 02, \qquad \tau_b = 34 \text{ s} \tag{8}$$

Moreover, the position and velocity at the time of first observation are

$$X = 3475.1 \text{ km}, \qquad Y = -1229.3 \text{ km}, \qquad Z = 5266.6 \text{ km}$$
$$\dot{X} = 1.120 \text{ km/s}, \qquad \dot{Y} = 0.658 \text{ km/s}, \qquad \dot{Z} = 1.284 \text{ km/s} \tag{9}$$

The missile parameters and state variables [Eqs. (8) and (9)] are unknown values that are estimated by the method described previously. In Table 1, the observation times t_k and τ_k, $k = 0, \ldots, N - 1$ $(\tau = t - t_0)$; the x, y, and z coordinates; the estimated average thrust-to-weight ratio $\bar{n}_k$; and the real thrust-to-weight ratio $n(t)$ are reported.

The true values of the thrust-to-weight ratio are zero after the burnout time; from the preceding result, one has $\bar{n}_4 < \bar{n}_3$, and so the estimated burnout time τ_b is in the interval of 30–40 (recall that the true burnout time is $t_b = 113$ s, that is, $\tau_b = 34$ s). Improving such an estimate according to the preceding method, one gets 33.5 s as the estimated burnout time. As it concerns the ballistic coefficient, Eq. (7) gives the values $\bar{\beta}_1 = 2.25e - 04$.

The performance of the method is evaluated considering the impact point prediction error. The trajectory, obtained by the previously estimated values after the first five radar measurements, is propagated numerically till the impact point. It turns out that the distance between the true and the estimated impact points is just 9 m.

III. Missile Detection from Measures Corrupted by Noise

The radar data are now affected by noise; the range, azimuth, and elevation measures are represented by

$$\tilde{\rho}_k = \rho_k + d_{k1}\sigma_\rho, \qquad \tilde{A}_{zk} = A_{zk} + d_{k2}\sigma_{A_z}$$
$$\tilde{E}_{lk} = E_{lk} + d_{k3}\sigma_{E_l} \tag{10}$$

In Eq. (10) σ_ρ, σ_{Az}, and σ_{El} are the range, azimuth, and elevation standard deviations, respectively, and the numbers d_{k1}, d_{k2}, and d_{k3} are random numbers chosen from a normal distribution with mean zero and a variance of one. Then a statistical method of prediction will be followed, based on the maximum likelihood estimate. Assuming the missile in gravity-turn phase, the method has to identify the state condition at time $\tau = \tau_1$

$$\boldsymbol{X}_1 = (X_0, Y_0, Z_0, \dot{X}_0, \dot{Y}_0, \dot{Z}_0) \tag{11}$$

and the fundamental missile parameters at time $\tau = \tau_1$

$$\boldsymbol{p}_1 = (\bar{n}_1, I_{sp}, \bar{u}_1, \beta_1) \tag{12}$$

so that the range, azimuth, and elevation values

$$\tilde{\rho}_k = \rho(\boldsymbol{X}_1, \boldsymbol{p}_1, t_k), \qquad \tilde{A}z_k = Az(\boldsymbol{X}_1, \boldsymbol{p}_1, t_k)$$
$$\tilde{E}l_k = El(\boldsymbol{X}_1, \boldsymbol{p}_1, t_k)$$

that are obtained by integrating the dynamic system [Eq. (1)] till time τ_k with data [Eqs. (11) and (12)] generate a minimum for the error function

$$\mathrm{Err}\,(\boldsymbol{X}_1, \boldsymbol{p}_1) = \sum_{k=1}^{N} \left\{ \frac{(\rho_k - \tilde{\rho}_k)^2}{2\sigma_\rho^2} + \frac{(Az_k - \tilde{A}z_k)^2}{2\sigma_{Az}^2} + \frac{(El_k - \tilde{E}l_k)^2}{2\sigma_{El}^2} \right\} \tag{13}$$

To find a minimum of the function [Eq. (13)], iterative algorithms are used that rely on a suitable initial guess for the first iteration data [Eqs. (11) and (12)]. It turns out that the estimate proposed in Sec. II provides a good set of guessed values. As an example, 400 tests are performed generating noised measures of the preceding missile trajectory. The standard deviations considered are $\sigma_\rho = 100$ m, $\sigma_{Az} = \sigma_{El} = 0.2°$, and random numbers d_{k1}, d_{k2}, and d_{k3} are generated 400 times. Guessed values for any of the 400 tests were generated with the algorithm of Sec. II. The performance of the method is evaluated considering the impact point prediction error based on the first five observations only. The 400 impact points define a 3σ-dispersion ellipse having semi-axis distances $a = 227.408$ km and $b = 2.625$ km.

Figure 1a shows distances of the impact points from the major axis a and from the minor axis b in the form of a histogram.

The figure shows that the error in impact point of the preceding medium-range (about 1800 km) missile trajectory is within 50 km for 47% of the tests. This seems an acceptable result, taking into account that this estimate is based on only five radar observations in the boost phase. Of course, accuracy can be greatly improved if some observations are at disposal in the ballistic phase.

Taking into account the 10 observations, it follows that in the 80% of the 400 tests, the burnout time estimate falls within the time interval $\tau \in [31\ \mathrm{s}, 34\ \mathrm{s}]$. In all the tests, it is recognized that the observation at time $\tau = 50$ sees the missile in ballistic phase. Then the observations after $\tau = 50$ s are processed by the preceding procedure, with error function depending on the (constant) ballistic coefficient $B = C_D[\beta_1/(1 - \bar{q}\tau_b)]$ and on the velocity initial condition only $[\mathrm{Err} = \mathrm{Err}(\dot{X}_0, \dot{Y}_0, \dot{Z}_0)]$. The dispersion ellipse determined by all the radar measurements at disposal (10 observations) has the major semi-axis distance $a = 26.563$ km and the minor semi-axis distance $b = 1.459$ km. The histogram of the distances from the minor and major semi-axes of the dispersion ellipse is shown in Fig. 1b.

IV. Real-Time Method: Extended Kalman Filter

The extended Kalman filter's main achievement with respect to MLE is the opportunity it has to be implemented as a real-time method, allowing the on-time tracking of observed objects. Two different EKFs have been implemented.

The first solution consists of a fully autonomous filter; the second possibility uses the results achieved by means of MLE in the boosted phase to initialize the filter.

The autonomous EKF uses an eight-state vector for the first part of trajectory, adding n_1 and q_1 to position and velocity components. The filter guess values for position and velocity were obtained by solving the classical Lambert theorem from the first and second measurements (thus, they are acquired during boosted phase). Starting n_1 and q_1 have been set to 2 and 10^{-5}, respectively.

The dynamic model [Eq. (1)], with two added equations to introduce the constants n_1 and q_1, has been used and written in recursive form. The covariance matrix P is propagated through

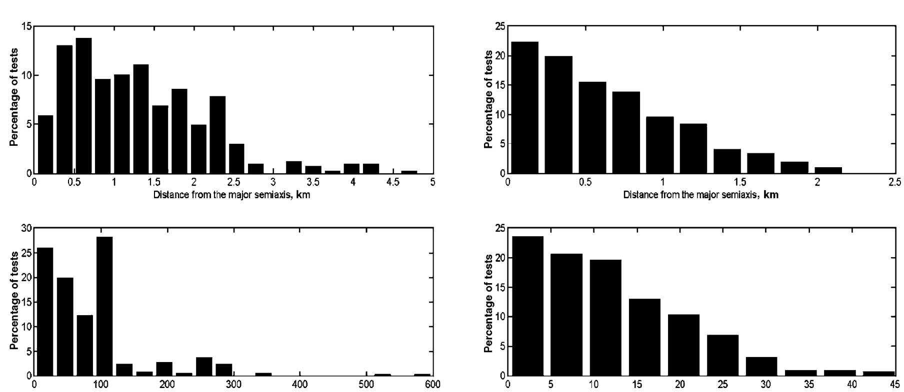
$$P_{k+1} = \phi_{k+1} P_k \phi_{k+1} + Q$$

where ϕ_k is the dynamic matrix at step k and Q is the matrix containing process noise. Additive process noise has been considered in the fourth, fifth, and sixth equations, and the derivatives are analytically evaluated. The matrix P_0 has been chosen diagonally, considering variances of 10 km on position components, 1 km/s for velocity components, 10 on n_1, and 10^{-4} on q_1. The drag force has been considered as process noise, in fact, at considered height and velocity, drag effect is rather small. The inertial components of position are considered in the filter and the Kalman gain has been evaluated from the well-known relationship

$$K_k = P_k H_k^T \left(H_k P_k H_k^T + R_k \right)^{-1}$$

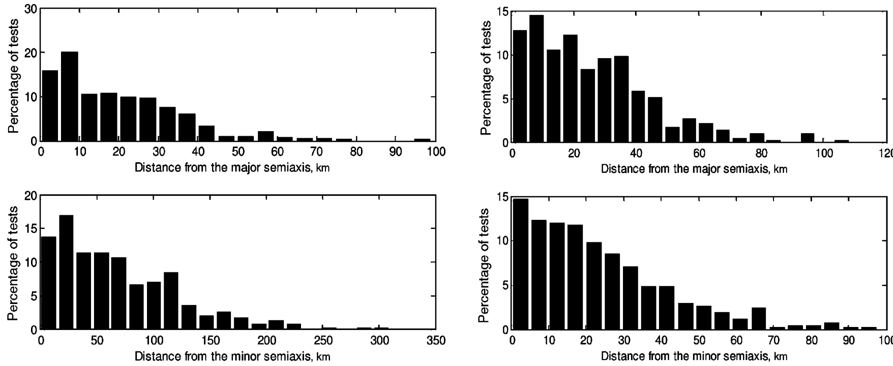
where H is the measures matrix that permits the correlation of measures to the state, and R is the measurement noise covariance matrix valuated considering the preceding accuracies on radar observables.

This filter can be used during the boosted phase. It is possible to identify the burnout interval by considering the covariance matrix



a) Five measurements b) Ten measurements

Fig. 1 Histograms of the distances of the impact points.



a) Autonomous EKF **b) EKF mixed with MLE**

Fig. 2 Histograms of the distances of the impact points; 10 measurements.

eigenvalues. In the burnout interval, the eigenvalues decreasing diminishes because of the differences between the model propagation and the measures. This procedure can be used to monitor the filter and switch from the boosted model to the ballistic one.

A different EKF starts after burnout interval. This part of the trajectory is treated as a ballistic phase, considering drag effect as process noise. The same procedure of the first part is used, but only position and velocity have been considered in the state vector.

Results achieved on example trajectory are depicted in Fig. 2a. The standard deviations associated to dispersion ellipse are 54.07 km for distances from the minor semi-axis and 15.66 km for distances from the major semi-axis. It has to be noted that in 2% of test cases, the algorithm has not been able to correctly individuate the burnout interval.

A second way to perform trajectory determination has been implemented. In this latter case, the EKF has been used only in the ballistic phase, exploiting results achieved from MLE in the boosted phase as the starting state vector.

The standard deviations associated with dispersion ellipse are 19.63 and 18.92 km, respectively, for distances from the semiminor axis and semimajor axes. The histogram of the distances from the minor and major semi-axes of the dispersion ellipse is shown in Fig. 2b.

V. Conclusions

A simplified analytic solution of the boost phase of a missile is applied here to estimate the four missile fundamental parameters and the six state variables. Such an analytical estimate provides a suitable guess for the convergence of a postprocessing algorithm based on the maximum likelihood estimate method. An extended Kalman filter has been applied to the same problem; in the present case (search radar with low frequency rate), better outcomes have been achieved implementing the filter, considering missile parameters obtained by maximum likelihood estimate.

Besides, a combined use of batch-detection algorithms is proposed here for the boosted phase and real-time detection algorithms for the ballistic and reentry phases.

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